An Example of J-unitary Operator. Solving a Problem Stated by M.G. Krein.

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Abstract

Theorem 1. Given a number $c \geq 1$, there exists a J-unitary operator $\widehat{V}$, such that:

(a) $r(\widehat{V}) = r(\widehat{V}^{-1}) = c$;

(b) $S(\widehat{V}, c) = S(\widehat{V}^{-1}, c) = S(\widehat{V}^{*-1}, c) = S(\widehat{V}^{*}, c) = \{0\}$.

(c) there exist maximal strictly positive and strictly negative $\widehat{V}^{\pm}$ − invariant subspaces $\mathcal{L}_+, \mathcal{L}_-$ such that they are mutually J-orthogonal and $\mathcal{L}_+ + \mathcal{L}_-$ is dense in the space.

(d_1) if L_1 is non-zero $\widehat{V}$-invariant subspace, $\widehat{V}L_1 \subset L_1$, then $r(\widehat{V}|L_1) = r(\widehat{V})$.

(d_2) if L_2 is non-zero $\widehat{V}^{-1}$-invariant subspace, $\widehat{V}^{-1}L_2 \subset L_2$, then $r(\widehat{V}^{-1}|L_2) = r(\widehat{V}^{-1})$.

In this paper we try to essencially refine some results of 0404071. So, the paper inherit the system of notations and definitions of 0404071. For the references, historical comments, etc., see 0404071 as well.

To specify the Problem, recall some definitions and facts

Definition D1-1 Given an operator T and a number c, we write

$$S(T, c) := \{x \in H \,|\, \exists M \geq 0 \,\forall N \geq 0 \quad \|T^N x\| \leq Mc^N\},$$

$$r(T) := \text{spectral radius of } T\,.$$

Remark R1-1 Given a linear bounded operator T and a T-invariant subspace L such that $r(T|L) \leq c$, then

$$L \subset S(T, c + \epsilon)$$

for any $\epsilon > 0$.

Remark R1-2

$$S(T_1 \oplus T_2, c) = S(T_1, c) \oplus S(T_2, c);$$

The central theorem of this paper is

Theorem 1. Given a number $c \geq 1$, there exists a J-unitary operator $\widehat{V}$, such that:

(a) $r(\widehat{V}) = r(\widehat{V}^{-1}) = c;$

(b) $S(\widehat{V}, c) = S(\widehat{V}^{-1}, c) = S(\widehat{V}^{*-1}, c) = S(\widehat{V}^{*}, c) = \{0\}\,.$

(c) there exist maximal strictly positive and strictly negative $\widehat{V}^{\pm}$ $-$ invariant subspaces $\mathcal{L}_+, \mathcal{L}_-$ such that they are mutually J-orthogonal and $\mathcal{L}_+ + \mathcal{L}_-$ is dense in the space.

(d_1) if L_1 is non-zero $\widehat{V}$-invariant subspace, $\widehat{V} L_1 \subset L_1\,,$ then $r(\widehat{V}|L_1) = r(\widehat{V})\,.$

(d_2) if L_2 is non-zero $\widehat{V}^{-1}$-invariant subspace, $\widehat{V}^{-1} L_2 \subset L_2\,,$ then $r(\widehat{V}^{-1}|L_2) = r(\widehat{V}^{-1})\,.$

Proof . By the Lemma 6 below there exists an operator V in a separable (real or complex, as desired) Hilbert space, such that V is bounded, V^{-1} exists, is bounded, and

$$r(V) = r(V^{-1}) = c$$

$$S(V, c) = S(V^{-1}, c) = S(V^{*-1}, c) = S(V^{*}, c) = \{0\}\,.$$

Now put

$$\widehat{V} = V \oplus V^{*-1}$$

and the proof of (a), (b) is evident. To prove (c) apply Lemma 7.
Now proof of (d_1).
Suppose $r(\widehat{V}|L_1) < r(\widehat{V})$. Put $\epsilon = r(\widehat{V}) - r(\widehat{V}|L_1)$. Then $\epsilon > 0$ and

$$L_1 \subset S(\widehat{V}, r(\widehat{V}|L_1) + \epsilon) = S(\widehat{V}, r(\widehat{V})) = \{0\}\,.$$

Contradiction with $L_1 \neq \{0\}$. Proof of (d_2) is quite similar.
□

Now the details.

Let $\{u_n\}_{n\in\mathbf{Z}}$ be a bilateral number sequence; we will suppose that $u_n \neq 0$ for all $n \in \mathbf{Z}$.

In this case let U denote the **shift** [1] that is generated by the formula

$$U \,:\, b_n \,\mapsto\, \frac{u_{n+1}}{u_n}\, b_{n+1} \qquad . \tag{$*$}$$

The general facts we need are these:

Observation O2-1

One constructs the U as follows:

One starts extending the instruction $(*)$ on the linear span of the $\{b_n\}_{n\in\mathbf{Z}}$ so that the resulted operator becomes linear. That extension is unique and defines a linear densely defined operator, which is here denoted by U_{min}, and which is closable. The closure of U_{min} is just the U.

Now then, this U is closed and at least densely defined and injective; it has dense range and the action of U^N, U^{*-N}, $U^{*N}U^N$, $U^{-N}U^{*-N}$ (for any integer N) is generated by

$$U^N : b_n \mapsto \frac{u_{n+N}}{u_n} b_{n+N}\,; \qquad U^{*-N} : b_n \mapsto \frac{u_n^*}{u_{n+N}^*} b_{n+N}\,;$$

$$U^{*N}U^N : b_n \mapsto |\frac{u_{n+N}}{u_n}|^2 b_n\,; \qquad U^{-N}U^{*-N} : b_n \mapsto |\frac{u_n}{u_{n+N}}|^2 b_n\,.$$

In particular, U^N is bounded just when the number sequence $\{|u_{n+N}/u_n|\}_n$ is bounded. Moreover,

$$\|U^N\| = sup\{\ |u_{n+N}/u_n|\ |n \in \mathbf{Z}\}$$

[1] the full name is: the **bilateral weighted shift of $\{b_n\}_n$, to the right**.

□

The special factors we need are :

Observation O2-2(revised) Let $a > 0$ be number. Then

$$
\begin{array}{lcrcll}
S(U,a) \neq \{0\} & \Leftrightarrow & \exists M^{'}\,|u_N| & \leq & M^{'}a^N & (N = 0,1,2,...) \\
S(U^{*-1},a) \neq \{0\} & \Leftrightarrow & \exists M^{'}\,|u_N|^{-1} & \leq & M^{'}a^N & (N = 0,1,2,...) \\
S(U^{-1},a) \neq \{0\} & \Leftrightarrow & \exists M^{'}\,|u_{-N}| & \leq & M^{'}a^N & (N = 0,1,2,...) \\
S(U^{*},a) \neq \{0\} & \Leftrightarrow & \exists M^{'}\,|u_{-N}|^{-1} & \leq & M^{'}a^N & (N = 0,1,2,...)
\end{array}
$$

Proof. The family $\{b_n\}_n$ is an orthonormal basis. In addition $U^N b_n \perp U^N b_m$ for $n \neq m$. Thus

$$\|U^N f\|^2 = \sum_n |(b_n, f)|^2 \|U^N b_n\|^2 = \sum_n |(b_n, f)|^2 |\frac{u_{n+N}}{u_n}|^2$$

for every $f \in D_{U^N}$.

In particular,

$$\|U^N f\| \geq |(b_n, f)||u_{n+N}/u_n|$$

for all integers n.

It follows that:

Given $f \in H_0 \setminus \{0\}$ and given some real M, a such that

$$\|U^N f\| \leq M a^N \text{ for } N = 0,1,2,\dots,$$

then there exists a real M' such that

$$|u_N| \leq M' a^N \text{ for } N = 0,1,2,...$$

On the other hand, if

$$\exists M^{'}\,|u_N| \leq M^{'} a^N \; (N = 0,1,2,...)$$

then $b_0 \in S(U,a)$ and $S(U,a) \neq \{0\}$.

For U^{*-1}, U^{-1}, U^*, the proof is similar.

Lemma 1.
Let

$$|v_n| \quad \neq \quad 0$$

and V be a bilateral weighted shift defined by

$$Vb_n \quad := \quad \frac{v_{n+1}}{v_n} b_{n+1}$$

Suppose that

$$\exists n_0 \geq 0$$

so that

$$\forall n \qquad |n| \geq n_0 \Rightarrow \left| \frac{v_{n+1}}{v_n} \right| \leq 1 \qquad\qquad (*)$$

Then

$$||V|| < \infty$$

and

$$r(V) \quad \leq \quad 1$$

As a consequence

$$r(V^*) = r(V) \quad \leq \quad 1$$

Proof.
By supposition,

$$|n| \geq n_0 \Rightarrow \left| \frac{v_{n+1}}{v_n} \right| \leq 1 \qquad\qquad (*)$$

Put

$$c_0 = max\{1, max\left\{ \left| \frac{v_{n+1}}{v_n} \right| \quad \Big| \quad n = -n_0, -n_0 + 1, \cdots, n_0 \right\}\}$$

Hence for every $n \in \mathbf{Z}$, $N > 0$

$$\left| \frac{v_{n+N}}{v_n} \right| = \left| \frac{v_{n+1}}{v_n} \right| \cdots \left| \frac{v_{n+N}}{v_{n+N-1}} \right| \leq {c_0}^{2n_0+1}$$

Apply the Observation O2-1 and obtain that for every $N > 0$

$$\|V^N\| \leq {c_0}^{2n_0+1}$$

Well known

$$r(V) \leq \|V^N\|^{1/N}\,, \qquad (N = 1, 2, \ldots)$$

Hence

$$r(V) \leq {c_0}^{(2n_0+1)/N}\,, \qquad (N = 1, 2, \ldots)$$

and

$$r(V) \leq 1\,.$$

□

Lemma 2.
Let

$$|v_n| \quad \neq \quad 0$$

and V be a bilateral weighted shift defined by

$$Vb_n \quad := \quad \frac{v_{n+1}}{v_n} b_{n+1}$$

Suppose that

$$\exists c > 0 \exists n_0 \geq 0$$

so that

$$\forall n \qquad |n| \geq n_0 \Rightarrow \left|\frac{v_{n+1}}{v_n}\right| \leq c \qquad (*)$$

Then

$$||V|| < \infty$$

and

$$r(V) \quad \leq \quad c$$

As a consequence

$$r(V^*) = r(V) \quad \leq \quad c$$

Proof
Put

$$w_n = \frac{v_n}{c^n}$$

Then

$$\left|\frac{w_{n+1}}{w_n}\right| = \left|\frac{v_{n+1}}{v_n}\right| \frac{1}{c}$$

Hence

$$\left|\frac{w_{n+1}}{w_n}\right| \leq 1 \qquad (|n| \geq |n_0|)$$

Let W be a bilateral weighted shift defined by

$$Wb_n \quad := \quad \frac{w_{n+1}}{w_n} b_{n+1}$$

By the Lemma 1

$$\|W\| < \infty\,, \qquad r(W) \le 1\,.$$

On the other hand

$$W = \frac{1}{c} V\,.$$

Hence

$$V = cW\,.$$

Hence

$$r(V) = c\, r(W) \le c\,.$$

□

Lemma 3
Let

$$|v_n| \quad \neq \quad 0$$

and V be a bilateral weighted shift defined by

$$Vb_n \quad := \quad \frac{v_{n+1}}{v_n} b_{n+1}$$

Suppose that

$$\exists n_1 \geq 0$$

so that

$$\forall n \qquad |n| \geq n_0 \Rightarrow \left| \frac{v_{n+1}}{v_n} \right| \geq 1 \qquad (*)$$

Then

$$||V^{-1}|| < \infty$$

and

$$r(V^{-1}) \quad \leq \quad 1$$

As a consequence

$$r(V^{-1*}) = r(V^{-1}) \quad \leq \quad 1$$

Proof
By supposition,

$$|n| \geq n_1 \Rightarrow \left| \frac{v_{n+1}}{v_n} \right| \geq 1$$

Hence

$$|n| \geq n_1 \Rightarrow \left| \frac{v_n}{v_{n+1}} \right| \leq 1$$

and then

$$|n-1| \geq n_1 \Rightarrow \left| \frac{v_{n-1}}{v_n} \right| \leq 1$$

$$|n| \geq n_1 + 1 \Rightarrow \left| \frac{v_{n-1}}{v_n} \right| \leq 1$$

Put

$$c_1 = max\{1, max\Big\{ \left| \frac{v_{n-1}}{v_n} \right| \quad \Big| \quad n = -n_1 - 1, -n_1, \cdots, n_1 + 1 \Big\}\}$$

Hence for every $n \in \mathbf{Z}$, $N > 0$

$$\left|\frac{v_{n-N}}{v_n}\right| = \left|\frac{v_{n-1}}{v_n}\right| \cdots \left|\frac{v_{n-N}}{v_{n-(N-1)}}\right| \leq {c_1}^{2n_1+3}$$

Apply the Observation O2-1 and obtain that for every $N > 0$

$$\|V^{-N}\| \leq {c_1}^{2n_1+3}$$

Well known

$$r(V^{-1}) \leq \|V^{-N}\|^{1/N}, \qquad (N = 1, 2, \ldots)$$

Hence

$$r(V^{-1}) \leq {c_1}^{(2n_1+3)/N}, \qquad (N = 1, 2, \ldots)$$

and

$$r(V^{-1}) \leq 1\,.$$

$\square$

Lemma 4
Let

$$|v_n| \quad \neq \quad 0$$

and V be a bilateral weighted shift defined by

$$Vb_n \quad := \quad \frac{v_{n+1}}{v_n} b_{n+1}$$

Suppose

$$\exists c \geq 0 \exists n_1 \geq 0$$

so that

$$\forall n \qquad |n| \geq n_1 \Rightarrow \left|\frac{v_{n+1}}{v_n}\right| \geq c^{-1} \qquad\qquad (**)$$

Then

$$||V^{-1}|| < \infty$$

and

$$r(V^{-1}) \quad \leq \quad c$$

As a consequence

$$r(V^{-1*}) = r(V^{-1}) \quad \leq \quad c$$

Proof.
Put

$$w_n = v_n c^n$$

Then

$$\left|\frac{w_{n+1}}{w_n}\right| = \left|\frac{v_{n+1}}{v_n}\right| c$$

Hence

$$\left|\frac{w_{n+1}}{w_n}\right| \geq 1 \qquad (|n| \geq |n_1|)$$

Let W be a bilateral weighted shift defined by

$$Wb_n \quad := \quad \frac{w_{n+1}}{w_n} b_{n+1}$$

By the Lemma 3

$$\|W^{-1}\| < \infty\,, \qquad r(W^{-1}) \le 1\,.$$

On the other hand

$$W = cV\,.$$

$$V^{-1} = cW^{-1}\,.$$

Hence

$$r(V^{-1}) = c\,r(W^{-1}) \le c\,.$$

$\square$

Lemma 5
Let

$$|v_n| \quad \neq \quad 0$$

$$v_n \quad = \quad v_{-n}$$

and V be a bilateral weighted shift defined by

$$Vb_n \quad := \quad \frac{v_{n+1}}{v_n} b_{n+1}$$

Suppose that

$$\exists c \geq 1 \exists n_0 \geq 0$$

so that

$$\forall n \qquad |n| \geq n_0 \Rightarrow \left| \frac{v_{n+1}}{v_n} \right| \leq c \qquad (*)$$

Then

$$||V^{-1}|| = ||V|| < \infty$$

and

$$r(V^{-1}) = r(V) \quad \leq \quad c$$

As a consequence

$$r(V^*) = r(V) = r(V^{-1}) = r(V^{-1^*}) \quad \leq \quad c$$

Proof
Define R by

$$Rb_n \quad := \quad b_{-n}$$

Then

$$R^{-1} = R, \qquad R^2 = I, \qquad ||R|| = 1 < \infty, \qquad ||R^{-1}|| = 1 < \infty$$

Consider $R^{-1}SR$

$$R^{-1}VRb_n = R^{-1}Vb_{-n} = R^{-1}\frac{v_{-n+1}}{v_{-n}}b_{-n+1} = R^{-1}\frac{v_{n-1}}{v_n}b_{-n+1} = \frac{v_{n-1}}{v_n}b_{n-1} = V^{-1}b_n$$

Hence

$$R^{-1}VR = V^{-1}$$

Now apply the Lemma 2.

Lemma 6
Let

$$c \quad \geq \quad 1$$

$$\begin{aligned} \phi(x) &:= x \sin(\frac{\pi}{2} \log_2(1+\log_2(1+x))), \qquad (x \geq 0) \\ \psi(x) &:= x^{1/2} \sin(\frac{\pi}{2} \log_2(1+\log_2(1+x))), \qquad (x \geq 0) \end{aligned}$$

$$v_n \quad := \quad c^{\phi(|n|)} e^{\psi |n|}$$

so that

$$v_n \quad = \quad \overline{v_n}$$

$$v_n \quad > \quad 0$$

$$v_n \quad = \quad v_{-n}$$

and V be a bilateral weighted shift defined by

$$V b_n \quad := \quad \frac{v_{n+1}}{v_n} b_{n+1}$$

Then

$$r(V^*) = r(V) = r(V^{-1}) = r(V^{-1^*}) \quad = \quad c$$

and

$$S(V,c) = \{0\}\,,\, S(V^{-1},c) = \{0\}\,,\, S(V^{*-1},c) = \{0\}\,,\, S(V^*,c) = \{0\}\,.$$

Proof

$$z \quad := \quad \frac{\pi}{2} \log_2(1+\log_2(1+x))$$

and note

$$\begin{aligned} \phi'(x) &= \sin(z) + x\cos(z)\frac{\pi}{2}\frac{1}{1+\log_2(1+x)}\frac{1}{1+x}\frac{1}{(ln2)^2} \\ &= \sin(z) + \cos(z)\frac{1}{1+\log_2(1+x)}\frac{x}{1+x}k, \qquad k = \frac{\pi}{2(ln2)^2} \end{aligned}$$

Hence

$$\phi'(0) \quad = \quad 0$$

and if

$$x \quad \geq \quad 2^{2k/\epsilon} - 1$$

then

$$\begin{aligned} |\phi'(x)| &\leq 1 + \frac{1}{(1 + 2k/\epsilon)} k \\ &\leq 1 + \epsilon/2 \end{aligned}$$

Hence if

$$n \quad \geq \quad 2^{2k/\epsilon} - 1$$

then

$$|\phi(|n+1|) - \phi(|n|)| \quad \leq \quad 1 + \epsilon/2$$

Now note that if $x \neq 0$ then

$$\begin{aligned} \psi'(x) &= \frac{1}{2x^{1/2}} \sin(z) + x^{1/2} \cos(z) \frac{\pi}{2} \frac{1}{1 + \log_2(1+x)} \frac{1}{1+x} \frac{1}{(ln2)^2} \\ &= \frac{1}{2x^{1/2}} \sin(z) + \cos(z) \frac{1}{1 + \log_2(1+x)} \frac{x^{1/2}}{1+x} k, \qquad k = \frac{\pi}{2(ln2)^2}. \end{aligned}$$

As a result,

$$\psi'(x) \to 0 \text{ as } x \to \pm\infty$$

and

$$\psi(|n+1|) - \psi(|n|) \to 0 \text{ as } n \to \pm\infty$$

Hence if

$$c > 1$$

then

$$\forall \epsilon > 0 \exists n_0 \geq 0$$

so that

$$\forall n \qquad |n| \geq n_0 \Rightarrow \left|\frac{v_{n+1}}{v_n}\right| \leq c^{1+\epsilon}$$

Then apply the Lemma 5 and obtain

$$||V^{-1}|| = ||V|| < \infty$$

and

$$\forall \epsilon > 0$$

$$r(V^*) = r(V) = r(V^{-1}) = r(V^{-1^*}) \quad \leq \quad c^{1+\epsilon}$$

Hence

$$r(V^*) = r(V) = r(V^{-1}) = r(V^{-1^*}) \quad \leq \quad c$$

The case $c = 1$ is obvious.

Now choose two sequences of integers defining them by

$$n_k := 2^{2^{1+4k}-1} - 1; \quad m_k := 2^{2^{3+4k}-1} - 1 \qquad (k = 1, 2, ...)\,.$$

Then n_k, $m_k \in \mathbf{N}$, $n_k \to +\infty, m_k \to +\infty$ (as $k \to +\infty$), and simultaneously

$$v_{n_k} = v_{-n_k} = c^{n_k} e^{\sqrt{n_k}}\,; \quad v_{m_k}^{-1} = v_{-m_k}^{-1} = c^{m_k} e^{\sqrt{m_k}}\,.$$

We see that no estimation of the form

$$|v_N| \leq M' c^N, \quad |v_{-N}| \leq M' c^N, \quad |v_N|^{-1} \leq M' c^N, \quad |v_{-N}|^{-1} \leq M' c^N$$

(for $N = 0, 1, \cdots$) is possible. On looking at the Observation **O2-2**, we see that

$$S(V, c) = \{0\}\,,\ S(V^{-1}, c) = \{0\}\,,\ S(V^{*-1}, c) = \{0\}\,,\ S(V^*, c) = \{0\}\,.$$

This is just what was to be proven.
□

Lemma 7

$$\begin{aligned}
\{\widehat{U}^N(b_0 \oplus b_0), \widehat{U}^M(b_0 \oplus b_0)\} &= 0, \quad (M \neq N) \\
\{\widehat{U}^N(b_0 \oplus b_0), \widehat{U}^N(b_0 \oplus b_0)\} &= 2(b_0, b_0) = 2 > 0 \\
\{\widehat{U}^N(b_0 \oplus -b_0), \widehat{U}^M(b_0 \oplus -b_0)\} &= 0, \quad (M \neq N) \\
\{\widehat{U}^N(b_0 \oplus -b_0), \widehat{U}^N(b_0 \oplus -b_0)\} &= -2(b_0, b_0) = -2 < 0 \\
\{\widehat{U}^N(b_0 \oplus b_0), \widehat{U}^M(b_0 \oplus -b_0)\} &= 0.
\end{aligned}$$

Definition

$$\begin{aligned}
L_+ &:= span\{\widehat{U}^N(b_0 \oplus b_0) | N \in \mathbf{Z}\} \equiv span\{(U^N b_0 \oplus U^{*-N} b_0) | N \in \mathbf{Z}\} \\
L_- &:= span\{\widehat{U}^N(b_0 \oplus -b_0) | N \in \mathbf{Z}\} \equiv span\{(U^N b_0 \oplus -U^{*-N} b_0) | N \in \mathbf{Z}\}
\end{aligned}$$

Then

(1) $\widehat{U}^{\pm 1} L_+ = L_+$

(2) $\widehat{U}^{\pm 1} L_- = L_-$

(3) $b_N \oplus 0 \in L_+ + L_- \quad (N \in \mathbf{Z})$
$0 \oplus b_N \in L_+ + L_- \quad (N \in \mathbf{Z})$

(4) if $\{L_+ + L_-, x\} = 0$ then $x = 0$

(5) $\overline{L_+ + L_-} = \widehat{H}$

(6) $\{L_+, L_-\} = \{0\}$

(7) $\{x, x\} > 0 \quad (x \in L_+ \backslash \{0\})$

(8) $\{x, x\} < 0 \quad (x \in L_- \backslash \{0\})$

(9) $L_+ \cap L_- = \{0\}$

Moreover,

(6′) $\{\overline{L_+}, \overline{L_-}\} = \{0\}$

(7′) $\{x, x\} > 0 \quad (x \in \overline{L_+} \backslash \{0\})$

(8′) $\{x, x\} < 0 \quad (x \in \overline{L_-} \backslash \{0\})$

(9′) $\overline{L_+} \cap \overline{L_-} = \{0\}$

If in addition U and U^{-1} are bounded, then

$$(1') \qquad \widehat{U}^{\pm 1}\overline{L_+} = \overline{L_+}$$
$$(2') \qquad \widehat{U}^{\pm 1}\overline{L_-} = \overline{L_-}$$

Proof is straightforward.

Remark

$$g \oplus h \in L_+$$

$$g \oplus h = \sum_n g(n)b_n \oplus \sum_n \frac{|u_0|^2}{|u_n|^2} g(n)b_n$$

$$g \oplus h = g \oplus \sum_n \frac{|u_0|^2}{|u_n|^2} b_n(b_n, g)$$

$$g \oplus h \in L_-$$

$$g \oplus h = \sum_n g(n)b_n \oplus - \sum_n \frac{|u_0|^2}{|u_n|^2} g(n)b_n$$

$$g \oplus h = g \oplus - \sum_n \frac{|u_0|^2}{|u_n|^2} b_n(b_n, g)$$

$\square$

$$g \oplus h \in L_+$$

$$g \oplus h = \sum_n g(n)b_n \oplus \sum_n \frac{|u_0|^2}{|u_n|^2} g(n)b_n$$

$$\begin{aligned}
(h, U^N g) &= (\sum_n \frac{|u_0|^2}{|u_n|^2} g(n)b_n, U^N \sum_m g(m)b_m) \\
&= (\sum_n \frac{|u_0|^2}{|u_n|^2} g(n)b_n, \sum_m \frac{u_{m+N}}{u_m} g(m)b_{m+N}) \\
&= \sum_m \frac{|u_0|^2}{|u_{m+N}|^2} \overline{g(m+N)} \frac{u_{m+N}}{u_m} g(m)) \\
&= \sum_m \frac{|u_0|^2}{\overline{u_{m+N}}} \overline{g(m+N)} \frac{1}{u_m} g(m))
\end{aligned}$$

Remark to the Lemma 1 + Lemma 2, Lemma 3 + Lemma 4, Lemma 5.
The statements look very naturally, and I think may be known. But I have not found.